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DETERMINING PLACEMENT OF SYSTEMS DEMARCATED BY HEMISPHERES

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Abstract: A main problem when placing systems demarcated by hemispheres is determining coverage. Each system covers an area that can be represented by a hemisphere. The problem is how to cover an arena, a given area, with a minimal number of systems. Determining placement of systems demarcated by hemispheres requires arranging the systems on the plane so that an optimal cover of the plane is obtained. Our approach follows a top-down, stepwise refinement of the problem into simpler subproblems; the combined solution of all the subproblems yields the solution of the original problem. We first show that the original problem can be transformed from three to two-and-a-half dimensions and then decomposed into three subproblems. The first subproblem is finding the tiling possessing the minimal overlap between the hemispheres, the second is determining which hemispheres comprise the cover of a given arena, and the third is finding how the arena should be placed on the tiling.

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1. Introduction

Determining the optimal placement of systems required to cover a given area

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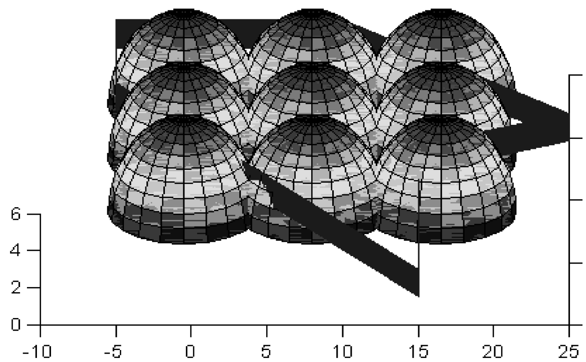

Figure 1: Minimal number of hemispheres of given identical size required to cover a given polyhedron

belongs to the classic point location geometric search problem. The optimal placement involves employing the minimum number of systems in the cover in order to minimize costs. The problem has civilian and military applications. Civilian applications include determining the minimum number of safety control systems, such as fire detection sensors, required to cover a given arena. Military applications include ascertaining the minimum number of defense systems required for aerial-ground coverage by missile, radar or ground sensors.

The systems can be represented by hemispheres of given identical size, while the area to be covered, the arena, can be represented as a polyhedron. The problem of covering a given area thus can be stated mathematically: what is the minimal number of hemispheres of given identical size required to cover a given polyhedron?

The following algorithm was developed to solve this problem. The solution first represents the three-dimensional problem in two-dimensions and then divides the problem into three subproblems: (1) What is the tiling possessing minimal overlap between the hemispheres? (2) How should the arena be placed on the tiling? and (3) Which hemispheres comprise the cover of the arena? Kershner's [7] optimal solution is used as the basis for finding the circle-based tiling possessing the minimal overlap. A geometric range search algorithm is then used to decide which hemispheres comprise the cover of the arena. A search method is presented for determining the placement of the arena on the tiling.

Prior related works [3, 4] assumed a fixed shape for the surface to be covered, for instance a rectangle, and a variety of shapes for the pieces to be placed. The contribution of this algorithm is that it focuses on the space to be covered, which is arbitrary. While the problem is more limited since the figures placed are identical hemispheres, the advantage is that it is not limited to a convex arena.

In Section 2 the problem is represented, and in Section 3 we define our terminology. Section 4 outlines the approach of the solution, and in the subsections the details are provided. The running time of the solution is discussed in Section 5, and in Section 6 we conclude with a summary and problems for further study.

2. Problem Representation

2.1. Representation in Three Dimensions

What is the minimal number of hemispheres of given identical size required to cover a given polyhedron? The hemispheres are positioned "sunny-side-up", so that the curvature of each hemisphere faces upward and overlapping is allowed among the hemispheres (Figure 1).

The arena, the area to be covered, can be represented as a polyhedron, with its faces representing the boundaries in the space to be covered. The individual areas covered are viewed as hemispheres situated on the polyhedron projection at the highest point in the plane.

In three dimensions, there are three periodic packings for identical spheres: cubic lattice, face-centered cubic lattice, and hexagonal lattice. Kepler conjectured in 1611 that close packing (cubic or hexagonal) is the densest possible (has the greatest packing density, which is the fraction of a volume filled by identical packed spheres). The problem of finding the densest packing of spheres (not necessarily periodic) is therefore the Kepler problem. The Kepler conjecture is intuitively obvious, but the proof remained elusive until it was accomplished in a series of papers by Hales [6] culminating in 1998. Gauss (1831) proved that the face-centered cubic is the densest lattice packing in three dimensions [2]. This result has been extended to hypersphere packing [10].

The problem is defined in a three-dimensional space, since the volume of the hemisphere covering the polyhedron determines the size of the arena protected. Hemispheres are used since they most accurately represent the area to be covered by the civilian system or defense system.

Each hemisphere represents a single three-dimensional system covering the space around it. The radius of the hemisphere demarcates the boundary of the system effectiveness. The systems must cover the entire arena without gaps between them; thus overlapping between the coverage area of the systems (the hemispheres) is necessary (see Figure 1). The coverage of the systems by hemispheres can extend over the border of the given arena. The polyhedron can represent a protected arena such as a room or field scanned by sensors or ground covered by systems. The polyhedron cannot be assumed to be convex, a fact which complicates finding a simple analytical solution.

2.2. Representation in Two Dimensions

The arena, the area to be covered by the hemispheres, is defined as the projection on the polyhedron when observing it from above. This perspective allows the problem to be represented in two dimensions. The projections of the hemispheres on the polyhedron are represented as circles.

3. Terminology

The following terms will be used:

Arena - The given polyhedron that is to be covered by the hemispheres.

Plane Tiling - A countable family of sets $\tau = T_1, T_2, \ldots$ which cover the plane without gaps or overlaps [5].

Tiles - The sets $T_1, T_2, \ldots$ of τ. The union of tiles is to be the whole plane, and the interiors of the sets T_i are to be pairwise disjoint [5].

Based on the definition of *Plane Tiling*, a new term will be used to limit the definition to a finite two-dimensional cover with equal overlapping spheres.

Arena Tiling: A finite number of tiles that cover the arena without gaps or overlaps.

4. Solution Outline

4.1. Transforming the Problem from Three to Two Dimensions

The problem of determining the minimal number of hemispheres with given radius needed to cover a given polyhedron is defined in a three-dimensional space. The arena is the space outlined by a non-convex polygon and a fence of height H surrounding the perimeter of the polygon. The arena is not necessarily placed on the ground but can be situated slightly above ground. The height of the arena above the ground represents the minimum operating height of the system.

The transformation from three dimensions to two provides an overhead view of the problem without any loss of information – in other words, the problem is transformed into $2\,\frac{1}{2}$ dimensions. In two dimensions the projection of the arena is represented by a polygon. The hemispheres of the systems are represented by circles that delineate the effectiveness border of the systems; the circles are the projections of the hemispheres on the projection of the polyhedron at its highest point. The center of each circle is the location of the system on the projection of the arena at the highest point.

The problem can now be divided into three subproblems in the two-dimensional space: (1) What is the tiling possessing minimal overlap between the hemispheres? (2) How should the arena be placed on the tiling? and (3) Which hemispheres comprise the cover of the arena?

4.2. Constructing a Two-Dimensional Plane Cover

This section builds a tiling that allows a two-dimensional cover of the plane to be constructed.

Kershner showed in 1939 that no arrangement of circles could cover the plane more efficiently than the hexagonal lattice arrangement shown in Figure 2 and defined by 6+1 overlapping circles, see [7]. The optimal solution describes an arrangement of spheres that covers the two-dimensional space and has the least overlap among the circles.

Once we have a two-dimensional plane cover with overlapping spheres as provided by the hexagonal lattice, the two-dimensional optimal plane cover will be used to find a basic tile that will then be used to cover the arena. Each tile vertex represents a two-dimensional sphere portraying a three-dimensional hemisphere. The hexagonal basic tile will be used to form the cover of the plane.

The basic tile of the hexagonal solution is a rhombus (a parallelogram with equal sides). Since each circle is surrounded by six other circles of a given radius, the values of each side of the rhombus is 2*radius*cos($\Pi/6$) and the values of the vertices are $\Pi/3$ and $2\,\Pi/3$.

Since the arena is of finite size, the cover must be large enough to include only the whole polygon. The algorithm creating a two-dimensional lattice using the basic tile is described in the following section. The algorithm shows the computations needed to determine the side of the rhombus. The lattice formed by the algorithm is displayed in Figure 2.

The tiling constructed above using the basic tile forms the floor on which the arena is to be placed. Each vertex of the tile represents a location of a

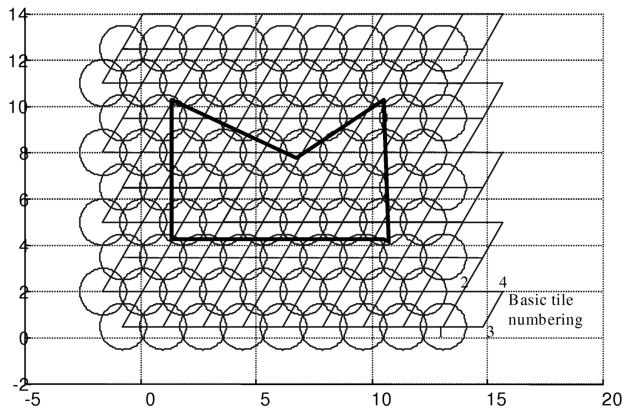

Figure 2: Lattice formed

system. The systems are optimally spaced to cover the maximum area without gaps between the system boundaries. The next step is determining how to place the arena on the floor in order to minimize the number of systems required.

4.3. The Hemispheres Covering the Arena

Once the polygon is placed on the lattice, the hemispheres forming the cover of the polygon must be determined and counted. This section builds an evaluation function to determine which hemispheres form the cover and to count them. This step is based on the locus method of attacking geometry problems [9], and on the optimal solution for a class point retrieval presented by Chazzelle, see [1]. The process of locating the hemispheres consists of two stages:

1. Mapping the hemispheres onto the X and Y axes while constructing the two-dimensional cover.

2. Parceling the lattice into the hemispheres containing the polyhedron.

4.3.1. Mapping the Hemispheres onto the X and Y Axis while Constructing the Two-Dimensional Cover

The basic tile used in the construction of the two-dimensional cover represents four hemispheres placed on the plane with their centers at the vertices and the distances between them. Thus each placement of the tile during the construction of the two-dimensional cover focuses on four hemispheres at a given moment. The vertices of a single tile will be numbered as shown in Figure 2.

The numbering of the vertices allows each hemisphere to be addressed separately.

4.3.2. The Storing Algorithm

The algorithm maps the hemispheres onto both the X and the Y axes and stores the location of the hemispheres, creating two linked lists arranged in ascending order, one for each axis. Each tile placement stores the location data of the hemispheres in the lists. Thus placement and ordering are done simultaneously. After all the tiles are placed, these two ordered lists enable us to locate the hemispheres placed between the maximum and the minimum of the X and Y coordinates of a line segment in two dimensions.

The algorithm is the same for both the X and Y lists. Each list stores the hemisphere location sorted according to either the X or the Y variable. There are three possible steps in the storing algorithm: initialization, moving up, and moving right. The *tile row* variable represents the number of the row currently being tiled in the algorithm.

Step Description	**Vertices Stored**
Initialization	First tile placed store vertices: 1, 2, 3 and 4 Set variable *tile row* = 1
Moving Up	Store vertices 1, 2 and 4 Vertex 3 is not stored: it was stored in the previous row, since it was the last vertex of the first tile in the row *tile row* = *tile row* + 1 since a new row has been started
Moving Right	If first *tile row* Store vertices 3 and 4 at the end of the list, since the two vertices determine the progress along the X axis Else Store vertex 4 since vertex 3 has been stored by the previous row.

4.3.3. Parceling the Lattice into the Hemispheres Containing the Polyhedron

After mapping the hemispheres onto the X and Y axes, the next step is to determine which hemispheres contain the arena polygon without *unnecessary* overlap between them. However, there can be excess circles in the cover determined. One instance describes the situation in which the circles could be covering the reflection of the edge of the polygon. Another instance describes unnecessary circles in the polygon cover, which can be formed by a line that connects two vertices and ends in the overlap between the two circles. The bottom circle in this example is unnecessary since the whole line is included in the top one. The final instance describes the internal circles included in the polygon cover but with no edge intersecting them. This instance allows a non-convex arena to be tiled by hemispheres.

4.3.4. The Algorithm for Parceling the Lattice into the Hemispheres Containing the Polyhedron

1. Map the hemispheres onto the X and Y axis (see prior algorithm).

2. Map the vertices of the arena polygon onto the X and Y axes using the same procedure that mapped the center of the circles. Each vertex receives an ordinal number. The numbering is performed by going over the arena polygon's vertices clockwise. Each arena vertex is placed between two centers of circles on each of the X and Y coordinate list.

3. For every two adjacent vertices, mark the circle centers between the maximum and minimum of the X and Y coordinates of the vertices as a possible arena polygon cover. Also add the centers of the circles adjacent to the circles in the cover to the *possible polygon cover*. Only centers of circles satisfying these conditions on both axes are marked as a possible part of the cover.

4. If the distance between the line connecting the two vertices and the circle center is greater than the radius, then the line is external to this circle. Remove the circle from the possible arena polygon cover.

5. If the distance between the line and the circle is greater than (2 cos 30 - 1)*R (subtract twice the radius from the distance between two adjacent circle centers), mark the circle as suspected for elimination from cover. For every circle suspected, check whether the intersection points with the line, including the edges of the line, are included in one of the other circles. If so, eliminate the circle.

6. Locate the intermediate circles included in the arena polygon cover by searching both the X and Y axis lists for circles that are surrounded by circles already belonging to the polygon cover. For each of these surrounded circles check the number of polygon edges crossed to reach the origin. If the number of lines is even then the circle is external to the polygon. Eliminate it from the arena polygon cover.

This completes the parceling of the lattice into the hemispheres that contain the polyhedron. The polygon cover now contains a list of all the hemispheres that contain the polyhedron. This list also completes the algorithm for finding the number of hemispheres covering a given polyhedron placement. The next and last step is to find the minimum number of hemispheres for any placement of the arena on the tiling.

4.4. The Placement of the Arena on the Tiling

The placement of the arena on the tiling consists of repetitive steps in which the hemispheres that form the cover of the polygon in each step are determined according to the algorithm described previously. The first part presents a search method for "all" possible placements of the arena on the tiling.

4.4.1. Pinball-Pin Method (PPM)

The pinball-pin method optimally places the arena on the plane tiled by circles. The algorithm describes the process of moving the polygon: it repetitively moves the polygon, first according to its edges, then according to its vertices, so that surplus circles can be removed from the cover and a minimal cover obtained. It is the movements that generate the optimal solution.

First, the polygon is moved parallel to each edge. As soon as an edge of the polygon intersects the circumference of a circle, i.e. is about to exit the cover, the polygon changes its direction of movement to that parallel to the position of the next edge. The movement of the polygon is similar to the movement of a pinball that bounces each time it hits a wall. Once the polygon has exited a circle as it is moved, the circle is no longer part of the cover. When the polygon has been moved according to all of the edges, the number of circles in the cover is calculated.

The next step is to pin down the polygon on each vertex and spin it a full 360 degrees. As the polygon is rotated about the vertex, the number of circles in the cover increases and decreases. Each time a circle is extracted from the cover, i.e. the polygon exits a circle, the number of circles in the cover is recalculated.

4.4.2. PPM Algorithm

1. Place the longest edge of the polygon so that it intersects the minimal number of circles.

2. Locate the edge that is of the next size down and move the polygon away from the edge until it intersects the circumference of a circle (Figure 3.A)

3. Repeat Step 2 for each edge.

4. Check the evaluation function for the number of circles covering the

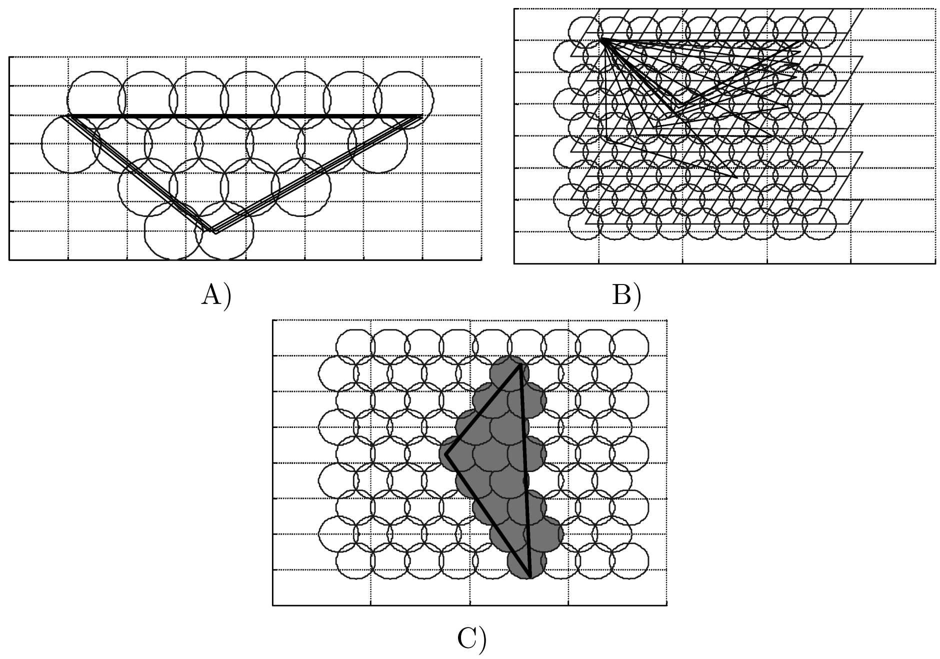

Figure 3: Pinball-pin method: Illustration

polygon (see the section: The hemispheres covering the arena).

5. For each vertex spin the polygon a full circle. For each circle extracted from the cover check whether the new cover yields a better result in the evaluation function (Figures 3.B, 3.C).

5. Analysis

The running time of the algorithm is $O(n^2)$, when n is the larger of two factors, the number of hemispheres or the number of polygon edges. The algorithm involves two linked lists of hemispheres, one for the X-axis and one for the Y, and we scan each of the two lists for every edge examined.

6. Summary

In this paper, an $O(n^2)$-time algorithm for determining the maximal coverage placement of systems demarcated by hemispheres is presented and illustrated.

The problem is transformed from three-dimensions to two-dimensions, and then the algorithm is based on three steps. First, we construct a two-dimensional cover of the plane. Second, we determine which hemispheres cover the arena and count them. Third, we place the arena on the tiling and determine its optimal placement.

The problem of constructing the tiling possessing the minimal overlap between the hemispheres is based on Kershner's [7] optimal solution. Finding the hemispheres comprising the cover of a given arena is based on a geometric range search algorithm. Determining the placement of the arena on the tiling employs a search method.

The algorithm has several limitations. First, although checking all possible cover solutions necessarily yields an optimal-minimal solution, this type of exhaustive search is not always most efficient. There is no need to check all intersection points of vertices and circles: if the two edges of a vertex are tangent to a circle, then the internal vertex cannot also be tangent to a circle that will delineate the cover of the arena. Thus, the PPM can be improved by eliminating these cases and searching only for all non-internal arena vertices.

Second, the figures placed are identical hemispheres, and not an assortment of different shapes of different sizes. Coverage of a certain arena might entail the use of systems of different coverage radii. Related problems that have not been addressed in this work might overcome this limitation. They include determining the minimal number of hemispheres of different sizes required to cover a given polyhedron, the minimal number of polyhedrons of identical size and shape required to cover a given polyhedron, and the minimal number of polyhedrons of different size and shape required to cover a given polyhedron. A possible direction of research is to attempt to combine the algorithm described in this work with the algorithms described in the articles of Dori and Ben-Bassat [3, 4], which arrange two-dimensional polygons within a rectangular board so that waste is minimized.

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